Mathematics Placement Packet
Colorado College Department of Mathematics and Computer Science

Colorado College has two all college requirements (Quantitative Reasoning, QR, and Scientific Investigations, SI) which can be satisfied in full, or part, by taking mathematics, statistics, or computer science courses. There are also numerous majors and minors at CC which require some calculus or statistics (see the bottom of this document for a list of such majors.) This worksheet and packet are designed to help you determine what might be a good first mathematics, or statistics. For computer science courses, please refer to the chart found here: `https://www.coloradocollege.edu/academics/dept/mathematics/where-to-start.html`.

Work your way through the decision tree on the next page. Consider the questions carefully and ask for advice or assistance if you are unsure. Use the following questions to help you as well. Note that the decision tree refers to four attached documents titled, *Sample Pre-Calculus, Sample Calculus I, Sample Calculus II*, and *Sample Calculus III.* Please look at and read these documents carefully, when appropriate.

1. Make a list of majors you are considering. This is not binding, just some majors you've been thinking about.

2. Are any of the majors on the list below? If yes, be sure to follow yes for "do you intend to study calculus at CC?" in the decision tree.

3. As you work through the decision tree, if you are stuck between two possible paths, consider the following questions:
   (a) When was the last time you took a math class?
   (b) How independent are you in your learning? Would you feel comfortable reviewing some topics on your own?

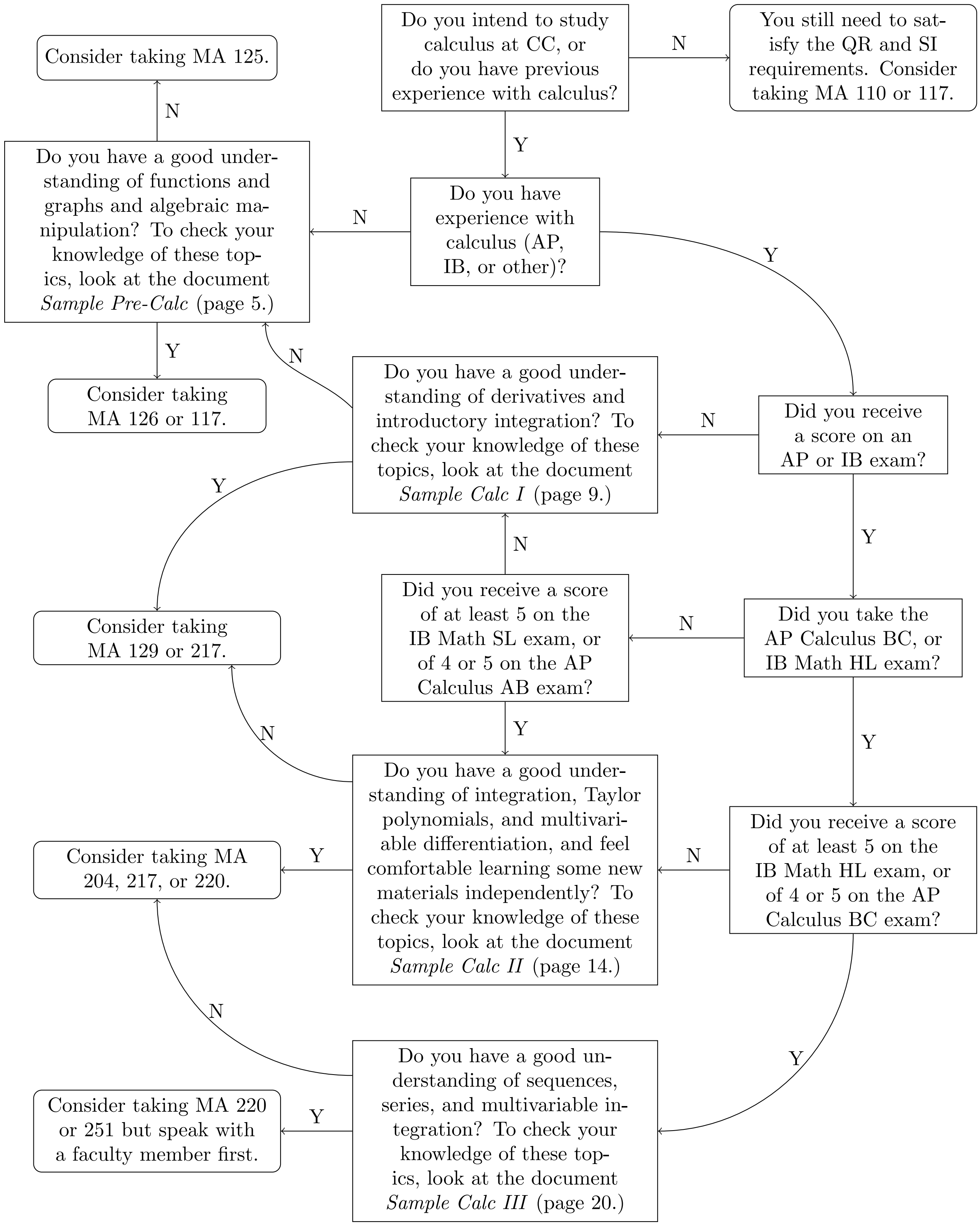
4. Write down the courses the tree suggested for you. Look these up on page 3 (*Choosing your First Math Class.*) After reading the corresponding course description, do these suggestions seem reasonable to you? If not, think about why not and talk to one of the Mathematics faculty.

To help answer the question, "**do you intend to study calculus at CC?**", here is a list of majors that require Calculus 1 (MA125 or MA126) or higher (some of these majors require significantly more mathematics.)

Molecular Biology, Organismal Biology & Ecology, Chemistry, Computer Science, Economics, Environmental Science, Geology, International Political Economy, Mathematics, Mathematical Economics, Physics, Psychology.

The following is a list of majors that require some statistics or modeling course from the list MA117, BY220, EC200, EV228, MA256, MA217. The specific course required varies across these majors, but MA117 satisfies most of these requirements (as will MA217).

Molecular Biology, Organismal Biology & Ecology, Economics, Environmental Science, Geology, International Political Economy, Mathematical Economics.


Do you intend to study calculus at CC, or do you have previous experience with calculus?
N
You still need to satisfy the QR and SI requirements. Consider taking MA 110 or 117.
Y
Do you have experience with calculus (AP, IB, or other)?
N
Do you have a good understanding of functions and graphs and algebraic manipulation? To check your knowledge of these topics, look at the document *Sample Pre-Calc* (page 5.)
N
Consider taking MA 125.
Y
Consider taking MA 126 or 117.
Y
Did you receive a score on an AP or IB exam?
N
Do you have a good understanding of derivatives and introductory integration? To check your knowledge of these topics, look at the document *Sample Calc I* (page 9.)
N
Y
Consider taking MA 129 or 217.
Y
Did you take the AP Calculus BC, or IB Math HL exam?
N
Did you receive a score of at least 5 on the IB Math SL exam, or of 4 or 5 on the AP Calculus AB exam?
N
Y
Y
Did you receive a score of at least 5 on the IB Math HL exam, or of 4 or 5 on the AP Calculus BC exam?
N
Do you have a good understanding of integration, Taylor polynomials, and multivariable differentiation, and feel comfortable learning some new materials independently? To check your knowledge of these topics, look at the document *Sample Calc II* (page 14.)
N
Y
Consider taking MA 204, 217, or 220.
Y
Do you have a good understanding of sequences, series, and multivariable integration? To check your knowledge of these topics, look at the document *Sample Calc III* (page 20.)
N
Y
Consider taking MA 220 or 251 but speak with a faculty member first.

Choosing Your First Math Class

If you took an AP or IB exam or a course at another university, you should receive official notification of credit and you are encouraged to take the next appropriate class. However, our calculus sequence is not traditional and you may find the following information helpful. Regardless, here are some rough guidelines on where to start, including some options for more advanced students or those wondering what might be a good next course. These descriptions are intended to enhance the Decision tree, which is a quick and easy place to start.

In mathematics, choose:
MA 125 (Pre-calc/Calc) if you need Calculus 1, but you need to review or learn the concepts from algebra and pre-calculus first. In particular, choose this course if you have trouble simplifying algebraic expressions involving exponents and fractions, if you have trouble seeing relationships between formulas and their graphs, or if you have not yet become acquainted with trigonometric and logarithmic functions. Check the document *Sample Pre-Calculus Questions* to help you with this decision.

MA 126 (Calc 1) if you have a solid algebra and pre-calculus background, but either have not yet taken calculus or do not feel comfortable with the Calculus I material.

MA 129 (Calc 2) if you successfully completed (and do not need help reviewing) a high school calculus course that covered techniques and applications of differentiation, and included a little introduction to integration, or if you completed calculus 2 (AP BC, for example), but did not score well on the exam, or do not feel comfortable with the Calculus II material.

MA 204 (Calc 3) if you successfully completed (and do not need help reviewing) a high school calculus 2 course (AP BC, for example), feel comfortable with that material and feel comfortable learning some new material independently.

MA 220 (Linear Algebra) if you successfully completed (and do not need help reviewing) a high school (or other) calculus 3 course that covered multivariable derivatives and integrals, alternate coordinate systems, and introductory vector calculus.

MA 256 (Mathematical Models in Biology) if you successfully completed (and do not need help reviewing) a high school (or other) Calculus I course that covered multivariable derivatives and integrals, alternate coordinate systems, and introductory vector calculus.

In statistics choose:
MA 117 (Probability & Statistics) if you have not had a calculus or statistics course.

MA 217 (Probability & Statistical Modeling) if you have successfully completed a calculus or statistics course, and like to see a little theory with your formulas. If you expect to be in a major that requires a substantial amount of statistics, you are strongly encouraged to consider this statistics course. If you have taken calculus (or higher) and feel comfortable working with the mathematical concepts, you are strongly encouraged to consider this statistics course.

Selecting a Quantitative Reasoning, QR, course: If you do not plan to continue in computer science, mathematics, the sciences (molecular biology, organismal biology & ecology, chemistry, environmental science, geology, physics, or psychology), or economics (including mathematical economics and international political economy), consider taking Math 110. This course typically has no prerequisites and is aimed at a general audience. A listing of Quantitative Reasoning courses currently offered can be found using the online course scheduling system.

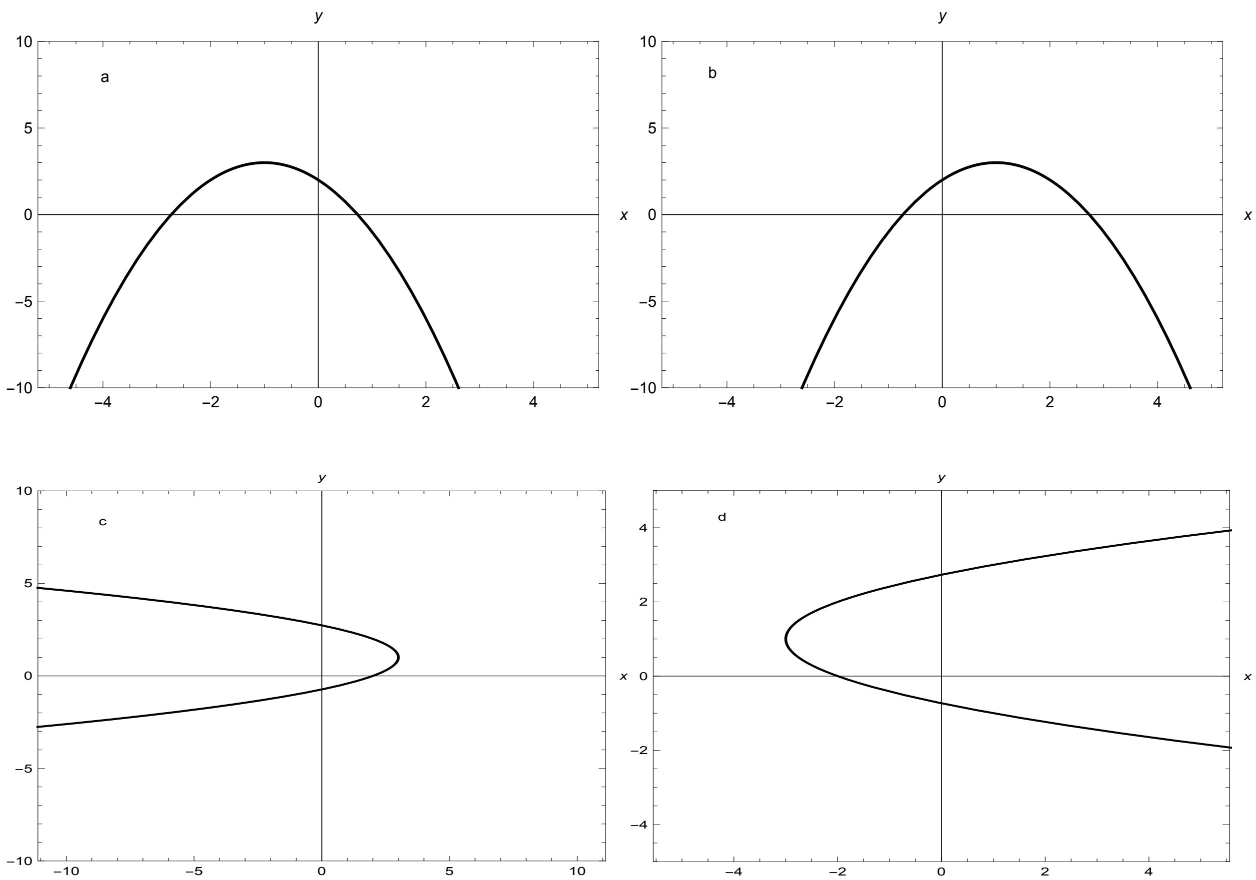
Sample Pre-Calculus Questions

A student need not get 100% on the questions below in order to take Math 126 rather than Math 125. An initial score of at least 50% is good enough as long as you feel that a little review would bring your score up to at least 80%. If you feel that you would need more than a little review, then you ought to enroll in Math 125.

1. Solve the equation $x^2 + 4x + 4 = 1$.

2. Find the equation of the line passing through $(2, -1)$ and parallel to $y = 4x + 5$.

3. Which of the following is the graph of $y = 3 - (x - 1)^2$?



4. Factor the following cubic polynomials, i.e. express each polynomial as a product of terms of the form $ax + b$:
   (a) $x^3 - x^2 - 6x$
   (b) $x^3 + 3x^2 - x - 3$

5. Suppose $2/3$ is 3 less than half of $4/5$ of some number. Find all such numbers.

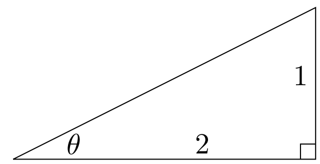
6. If $f(x) = 3x^2 - 5$, what is $f(-2)$, and $f(x + 1)$?

7. Assume that all fractions below are defined. That is, assume that the denominator is always non-zero. For each of the expression, determine if the right hand side is the simplified version

of the left hand side, i.e. determine if the equation, as stated, is true for all values of the variable(s) involved:

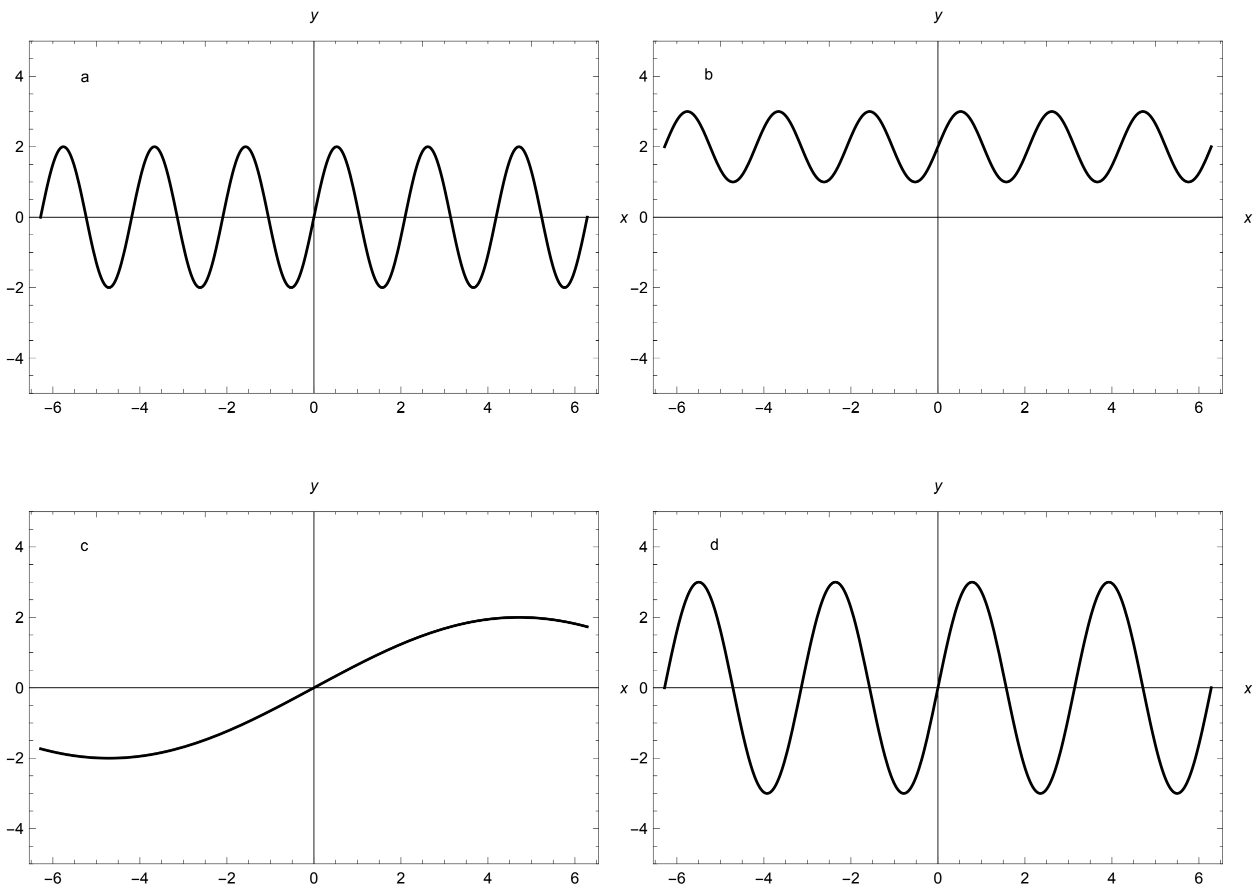
$$(I)\frac{x^2+1}{x} = x+1 \qquad (II)\frac{a^2+b}{b} = 1+\frac{a^2}{b} \qquad (III)\frac{y^2}{y+3} = y+\frac{y^2}{3}$$

8. Given the figure on the right, find the value of $\cos\theta$.



9. If $\sin\theta = \frac{\sqrt{3}}{2}$, what is $\tan\theta$?

10. Which of the following is the graph of $y = 2\sin(3x)$?



11. Find all solutions to each of the following equations
    (a) $2^x = 100$,
    (b) $3\log_2 x - \log_2 4 = 1$.

Pre-Calculus Answer Key

Make sure you not only get the correct answers but also provide sufficient justification.

1. We have

$$\begin{aligned} x^2 + 4x + 4 &= 1 \\ x^2 + 4x + 3 &= 0 \\ (x+1)(x+3) &= 0. \end{aligned}$$

   Thus, $x = -1$ or $x = -3$.

2. Parallel lines have the same slope so the slope of the line in question is 4. Now, using the slope-point formula, we have

$$y - (-1) = 4(x-2) = 4x - 8$$

   so the equation of the required line is $y = 4x - 9$.

3. Start with the graph of $y = x^2$, which is a (vertical) parabola. Shift this graph to the right by 1 unit to obtain the graph of $y = (x-1)^2$. Reflecting this new graph across the x-axis gives the graph of the line $y = -(x-1)^2$. Finally, shift the newly obtained graph by 3 units upward gives us the graph of $y = 3 - (x-1)^2$. Thus, the answer is (b).

   Alternatively, recognize the shape of the graph, and check some special points such as $x-, y-$ intercepts.

4. (a) $x^3 - x^2 - 6x = x(x^2 - x - 6) = x(x-3)(x+2)$

   (b) $x^3 + 3x^2 - x - 3 = x^2(x+3) - (x+3) = (x+3)(x^2-1) = (x+3)(x-1)(x+1)$.

   Note that when solving equations involving polynomials or factoring, the difference of squares formula $a^2 - b^2 = (a-b)(a+b)$ is often useful. Another useful approach is to see if terms can be grouped into pairs with common factor (like in part (b)). The last resort, if inspection yields no clear first step, is to note that for each polynomial $p(x)$, if a is a number such that $p(a) = 0$ then $p(x)$ can be expressed as a product of $(x-a)$ and a polynomial $q(x)$ of degree 1 smaller than the degree of $p(x)$.

5. Let the number is x. Then, we need

$$\frac{2}{3} < \frac{1}{2} \times \frac{4}{5}x - 3$$

   so $x > \left(\frac{2}{3} + 3\right) \times 2 \times \frac{5}{4} = \frac{55}{6}$. Thus, any number greater than $\frac{55}{6}$ satisfies the condition.

6. $f(x+1)$ is obtained by replacing x in the definition of $f(x)$ by $x+1$ so we have

$$f(x+1) = 3(x+1)^2 - 5 = 3(x^2 + 2x + 1) - 5 = 3x^2 + 6x - 2.$$

7. In this question, a valid equation means that the expression holds true for any arbitrary value(s) of the variable(s). Let's work out the expression on the left of the first two "equations":

$$\begin{aligned} \frac{x^2+1}{x} &= \frac{x^2}{x} + \frac{1}{x} = x + \frac{1}{x} \\ \frac{a^2+b}{b} &= \frac{a^2}{b} + \frac{b}{b} = \frac{a^2}{b} + 1. \end{aligned}$$

The expression of the left in (III) cannot be simplified in the same way because the sum is in the denominator. We can tell that (III) is not a true equation by choosing $y = 1$. Comparing what we obtained above to the expressions on the right-hand side, we find that only II holds.

8. Using right-angled triangle definition of cos, we know that it is the ratio of the adjacent side to the hypotenuse. Using Pythagorean theorem, the hypotenuse is $\sqrt{2^2+1^2} = \sqrt{5}$. Thus, $\cos\theta = \dfrac{2}{\sqrt{5}}$.

9. Using the trig identity $\sin^2\theta + \cos^2\theta = 1$, we get $\cos^2\theta = 1 - \dfrac{3}{4} = \dfrac{1}{4}$. This means $\cos\theta = \frac{1}{2}$ or $\cos\theta = -\frac{1}{2}$. (Recall that solutions to the equation $x^2 = a$, where $a > 0$ are $x = \sqrt{a}$ or $x = -\sqrt{a}$ instead of just $x = \sqrt{a}$.) Since $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, we have $\tan\theta = \sqrt{3}$ or $\tan\theta = -\sqrt{3}$.

   Alternatively, draw a right-angled triangle and use Pythagorean theorem to compute the value of the adjacent side. Make sure to think about to which quadrants θ can belong.

10. Start with the graph of $y = \sin x$. Compress the graph horizontally by a factor of 3 so that the interval $[0, 2\pi]$ comprises 3 full cycles. This produces the graph of $y = \sin(3x)$. Finally, stretch the new graph vertically by a factor of 2 to obtain the graph of $y = 2\sin(3x)$. The answer is (a).

11. (a) Recall the equivalence $y = b^x \iff \log_b y = x$ so $x = \log_2 100$.

    (b) Review different properties of logarithmic functions. We have

$$\begin{aligned}
&2\log_2 x - \log_2 4 = 1\\
&\log_2 x^3 - \log_2 4 = 1\\
&\log_2 \frac{x^3}{4} = 1\\
&\frac{x^3}{4} = 2^1 = 2\\
&x^3 = 8 \text{ so } x = 2.
\end{aligned}$$

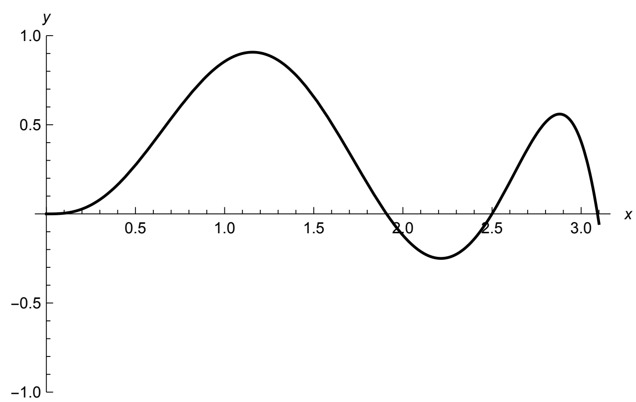
    Note that the steps above are not the only way to solve this equation.

Sample Calculus I Questions

A student need not get 100% on the questions below in order to skip Calculus I (Math 126) and go straight into Calculus II (Math 129). An initial score of at least 50% is good enough as long as you feel that a little review would bring your score up to at least 80%. If you feel that you would need more than a little review, then check the Pre-Calculus questions to determine if you should enroll in Math 126 or Math 125.

1. **Derivatives Symbolically**
   (a) Find the derivatives of the following functions.
      i. $y = x^3 + 2x$
      ii. $y = \cos(7x) + \dfrac{2}{x} + 6$
      iii. $y = (2x^4 + 7x)^{1/3}$
      iv. $y = e^{x^2}$
      v. $y = x \sin x$
      vi. $y = \ln(x^2 + 7)$
   (b) Let $f(x) = 2x^3 - 9x^2 + 12x + 1$.
      i. Find all intervals on which f is increasing.
      ii. Find all intervals on which f is concave up.
      iii. Find all local maxima for f.

2. **Derivatives Graphically** Let $f(x)$ be a function. The graph of its **derivative**, $f'(x)$, is shown below.



   Give approximate answers to the questions below.
   (a) On what interval(s) is f increasing?
   (b) On what intervals is f concave up?
   (c) At what point(s) does f reach a local maximum?
   (d) At what point(s) does f' reach a local maximum?
   (e) Is $f''(1.5)$ greater than 1 or less than 1?

3. **Derivatives Numerically**
   (a) Some numerical data is given for a function below.

x	0.0	1.0	1.7	2.3	2.5	3.5	4.2
$f(x)$	2.0	3.7	5.0	6.4	7.7	8.9	10.0

      i. What is the average rate of change of f on the interval $[0, 4.2]$?

ii. Estimate the instantaneous rate of change of f at $x = 3.5$ (there is more than one way to do this).

(b) The function $g(x)$ satisfies $g(3) = 10$, and $g'(3) = 2$. Estimate the value of $g(3.1)$.

4. **Derivatives Theoretically**

   Provide the limit definition of $f'(x)$, the derivative of f at x.

5. **Applications**

   (a) Find the equation of the line tangent to the graph of $f(x) = 3x^3 + \ln x$ at $x = 1$.

   (b) A commercial nursery has 1000 yards of fencing which the owners want to use to enclose as large a rectangular garden as possible. The garden will be bounded on one side by a barn, so no fencing is needed on that side. How large will the garden be (in square yards?)

6. **A Little Bit of Integration**

   (a) Find the following definite and indefinite integrals

   i. $\displaystyle\int (x^3 + 2x)\,dx$

   ii. $\displaystyle\int_1^2 (x^3 + 2x)\,dx$

   (b) Use the left-endpoint approximation with 4 equal subintervals to **estimate** the value of $\displaystyle\int_0^2 x^2\,dx$. (Do not evaluate the integral exactly using the Fundamental Theorem of Calculus.)

   (c) Find the general antiderivative of $f(t) = t^4 + 3\sin t$.

Calculus I Answer Key

Make sure you not only get the correct answers but also provide sufficient justification.

1. (a) i. $y' = 3x^2 + 2$

   ii. $y' = -7\sin(7x) - \dfrac{2}{x^2}$

   iii. $y' = \dfrac{1}{3}(2x^4 + 7x)^{-2/3}(8x^3 + 7) = \dfrac{8x^3 + 7}{3(2x^4 + 7x)^{2/3}}$

   iv. $y' = 2xe^{x^2}$

   v. $y' = \sin x + x\cos(x)$

   vi. $y' = \dfrac{2x}{x^2 + 7}$

   (b) i. We first find the first derivative of f: $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$. Setting $f'(x) = 0$ gives $x = 1$ or $x = 2$. Recall that if $f'(x) > 0$ on (a, b) then f is increasing on (a, b), and if $f'(x) < 0$ on (a, b) then f is decreasing on (a, b).
   Note that the values $x = 1, 2$ split the real number line into 3 intervals. We now look at the sign of $f'(x)$ on each of these intervals by testing $x = 0, 1.5, 3$:

$f'(x)$	$+$	1	$-$	2	$+$

   Alternatively, since the graph of $f'(x)$ is a parabola opening upward, we get $f'(x) > 0$ on $(-\infty, 1) \cup (2, \infty)$ and $f'(x) < 0$ on $(1, 2)$.
   Since f is continuous everywhere, f is increasing on $(-\infty, 1]$ and $[2, +\infty)$ and decreasing on $[1, 2]$. Note that continuity of f allows us to include the endpoints for the intervals over which f is increasing/decreasing.

   ii. The graph of a twice-differentiable function f is concave up on (a, b) if $f''(x) > 0$ on (a, b). Since $f''(x) = 12x - 18 = 6(2x - 3)$, the graph of f is concave up on $(3/2, \infty)$.

   iii. Using the information from part (i), we conclude that f has a local maximum at $x = 1$.

2. (a) When a function f is differentiable, f is increasing when $f'(x) > 0$ so f is increasing on $[0, 1.9]$, and $[2.5, 3.08]$.

   (b) When $f''(x) > 0$, the graph of f is concave up. In addition, $f''(x) > 0$ where f' is increasing. Thus, the graph of f is concave up on $(0, 1.15)$ and $(2.2, 2.9)$.

   (c) The function f has a local maximum at $x \approx 1.9$ and $x \approx 3.08$.

   (d) The function f' has a local minimum at $x \approx 1.15$ and $x \approx 2.9$.

   (e) Note that $f''(1.5)$ is the slope of the tangent line to the graph of f' at $x = 1.5$. Looking at the graph of f', we see that the tangent line at $x = 1.5$ has negative slope so $f''(1.5) < 1$.

3. (a) i. Recall that the average rate of change of a function f over an interval $[a, b]$ is given by $\dfrac{f(b) - f(a)}{b - a}$. Thus, the average rate of change of f on $[0, 4.2]$ is

   $$\frac{f(4.2) - f(0)}{4.2 - 0} = \frac{10.0 - 2.0}{4.2} = \frac{8.0}{4.2} = \frac{40}{21}.$$

ii. One way to estimate the instantaneous rate of change is to take the average of the average rates of change over intervals to the left and right of the point.

The average rate of change of f over $[2.5, 3.5]$ is $\dfrac{8.9-7.7}{3.5-2.5} = 1.2$. The average rate of change of f over $[3.5, 4.2]$ is $\dfrac{10.0-8.9}{4.2-3.5} = \dfrac{11}{7}$.

The estimated instantaneous rate of change of f at $x = 3.5$ is $\dfrac{1.2+\frac{11}{7}}{2} = \dfrac{97}{70}$.

(b) Recall that the linear approximation $L(x)$ to a function $f(x)$ near $x = a$ is given by

$$L(x) = f(x) + f'(a)(x-a).$$

Thus, the estimated value of $g(3.1)$ is

$$g(3.1) \approx L(3.1) = g(3) + g'(3)(3.1-3) = 10 + 2 \times 0.1 = 10.2.$$

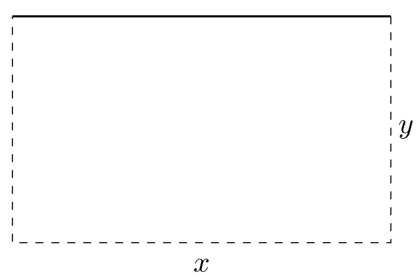
4. The derivative $f'(x)$ is defined to be the limit of the slope of the secant line so

$$f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}.$$

5. (a) We have $f'(x) = 9x^2 + \dfrac{1}{x}$ so the slope of the tangent line to the graph of f at $x = 1$ is $f'(1) = 9 + 1 = 10$. The equation of the tangent line is then

$$y = f(1) + 10(x-1) = 3 + 10(x-1) = 10x - 7.$$

(b)



The figure on the right shows the garden, where dashes represent fencing while the solid line represents the part of the barn that borders the garden. Let x, y be the dimensions, in yards, of the garden, as shown.

To maximize the area of the garden, we will use all of the fencing so that $x + 2y = 1000$ yards. This gives $x = 1000 - 2y$ yards.

The area of the garden is then $A(y) = (1000-2y)y = 1000y - 2y^2$ square yards. For the garden to make physical sense, we have $x, y > 0$. The restriction $x > 0$ then gives $1000 - 2y > 0$ so $y < 500$. The domain for A is thus $(0, 500)$.

We find critical point(s) of A by solving

$$0 = A'(y) = 1000 - 4y,$$

which gives $y = 250$ yards. The function A has only 1 critical point. In addition, $A'(y) > 0$ on $(0, 250)$ and $A'(y) < 0$ on $(250, 500)$. Since A is continuous on its domain, we conclude that A is increasing on $(0, 250]$ and decreasing on $[250, 500)$. Thus, A attains its absolute maximum at $y = 250$. (The barn is awfully big!) The largest area of the garden is

$$A(250) = (1000-500) \times 250 = 500 \times 250 = 125000 \text{ square yards.}$$

6. (a) i. $\int (x^3 + 2x)\,dx = \frac{x^4}{4} + x^2 + C$,

ii. By the Fundamental Theorem of Calculus, we have

$$\int_1^2 (x^3 + 2x)\,dx = \left.\frac{x^4}{4} + x^2\right|_1^2 = \frac{2^4}{4} + 2^2 - \left(\frac{1^4}{4} + 1^2\right) = 8 - \frac{5}{4} = \frac{27}{4}.$$

(b) Let $f(x) = x^2$. Since we are using 4 equal subintervals, $\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$. Then,

$$\begin{aligned}\int_0^2 x^2\,dx \approx L_4 = \sum_{i=0}^{3} f(0 + i\Delta x)\Delta x &= 0.5\,(f(0) + f(0.5) + f(1) + f(1.5)) \\ &= 0.5(0 + 0.25 + 1 + 2.25) = 1.75.\end{aligned}$$

(c) The general antiderivative of $f(t)$ is $F(t) = \frac{t^5}{5} - 3\cos t + C$, where C is a constant.

Sample Calculus II Questions

A student need not get 100% on the questions below in order to skip Calculus II (Math 129) and go straight into Calculus III (Math 204) or Linear Algebra (Math 220). An initial score of at least 50% is good enough as long as you feel that a little review would bring your score up to at least 80%. If you feel that you would need more than a little review, then you ought to enroll in Math 129.

1. **Integration Techniques and the Fundamental Theorem of Calculus**
   (a) Compute the following indefinite integrals.
      i. $\displaystyle\int x^2(5x^3+7)^6\,dx$
      ii. $\displaystyle\int x^3\ln(x^7)\,dx$
      iii. $\displaystyle\int xe^{x^2}\,dx$
      iv. $\displaystyle\int xe^{x}\,dx$

   (b) Suppose that the function $f(x)$ is defined by $f(x)=\displaystyle\int_1^x \frac{e^t}{t}\,dt$. Find $f'(x)$.

2. **Anti-derivatives Numerically**

   The Dundas city engineers have collected some data concerning the rate $r(t)$ of energy consumption per hour in their town. The data below represents a "typical" day in Dundas where the rate is measured in megawatts per hour, t hours after midnight.

t	0	4	8	12	18	20	24
$r(t)$	2.00	3.98	5.00	4.00	5.01	3.97	2.00

   Find a good estimate of the total 24 hour consumption of electricity in Dundas.

3. **Applications of Integration**
   (a) Compute the area bounded by the curves $y=6x$ and $y=3x^2$.
   (b) Rotate the region bounded by the x-axis, the curve $y=x^2$, and the line $x=3$ about the x-axis and find the volume of the resulting solid.
   (c) Solve the initial value problem $\dfrac{dy}{dt}=ye^{-t}, y(0)=1$.

4. **Improper Integrals**

   Which of the following improper integrals converge?
   (a) $\displaystyle\int_1^\infty \frac{1}{x}\,dx$
   (b) $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$
   (c) $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$
   (d) $\displaystyle\int_1^\infty e^{-x}\,dx$

5. **Vectors**

   Consider the vectors $\boldsymbol{v}=\langle 1,2,3\rangle$ and $\boldsymbol{w}=\langle -1,0,1\rangle$.
   (a) Compute the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$ in radians.
   (b) Compute the projection of the vector $\boldsymbol{v}$ onto $\boldsymbol{w}$, and find the orthogonal decomposition of $\boldsymbol{v}$. Draw a picture of the projection vector and the orthogonal decomposition of $\boldsymbol{v}$. (A 2D picture suffices.)

6. **Lines and Planes**

   (a) Determine whether the plane $3x - y + 2z = 7$ and the line $r(t) = \langle 1, 2, 1 \rangle + t\langle -2, 0, 1 \rangle$ intersect. If they do, find the point of intersection.

   (b) Find an equation for a plane that is perpendicular to both of the planes $x + y = 3$ and $x + 2y - z = 4$.

   (c) Find an equation of the tangent plane to the graph of $f(x, y) = 2x + 4y^2$ at the point $P = (-1, 2)$. Leave your answer in the form $ax + by + cz = d$.

7. **Extreme Values**

   Let $f(x, y) = (x^2 + y^2)e^{-x}$. Find all local extreme values and saddle points of f.

Calculus II Answer Key

Make sure you not only get the correct answers but also provide sufficient justification.

1. (a) i. Let $u = 5x^3 + 7$. Then, $du = 15x^2 dx$. Thus,

$$\int x^2(5x^3+7)^6\,dx = \frac{1}{15}\int (5x^3+7)^6(15x^2)\,dx = \frac{1}{15}\int u^6\,du = \frac{u^7}{7}+C = \frac{(5x^3+7)^7}{7}+C.$$

ii. We use integration by parts

$$\int x^3\ln(x^7)\,dx = \frac{x^4}{4}\ln(x^7) - \int \frac{x^4}{4}\frac{1}{x^7}(7x^6)\,dx = \frac{1}{4}x^4\ln(x^7) - \frac{7}{4}\int x^3\,dx$$
$$= \frac{1}{4}x^4\ln(x^7) - \frac{7}{16}x^4 + C.$$

Alternatively, simplify the integrand first before integrating by parts:

$$\int x^3\ln(x^7)\,dx = \int 7x^3\ln(x)\,dx = \frac{7x^4}{4}\ln(x) - \int \frac{7x^4}{4}\frac{1}{x}\,dx = \frac{7}{4}x^4\ln(x) - \frac{7}{4}\int x^3\,dx$$
$$= \frac{7}{4}x^4\ln(x) - \frac{7}{16}x^4 + C.$$

iii. Let $u = x^2$ so $du = 2xdx$. Then,

$$\int xe^{x^2}\,dx = \frac{1}{2}\int e^{x^2}(2x\,dx) = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u + C = \frac{1}{2}e^{x^2} + C.$$

iv. We use integration by parts in this case:

$$\int xe^x\,dx = xe^x - \int e^x\,dx = xe^x - e^x + C.$$

(b) Since $\dfrac{e^t}{t}$ is continuous for all $t \neq 0$, for any $x > 0$, the Fundamental Theorem of Calculus gives us

$$f'(x) = \frac{d}{dx}\int_1^x \frac{e^t}{t}\,dt = \frac{e^x}{x}.$$

Note that the function $f(x)$ is only defined for $x > 0$ since the lower limit of integration is 1. If $x < 0$, then we would be integrating across the point of discontinuity of the integrand.

2. The total energy consumption over the 24-hour interval is given by $\int_0^{24} r(t)\,dt$. Given the information, we use the trapezoid rule to estimate this integral and get:

$$\int_0^{24} r(t)\,dt \approx \frac{1}{2}[2.00 + 3.98 + 3.98 + 5.00 + 5.00 + 4.00 + 4.00 + 5.01 + 5.01 + 3.97 + 3.97 + 2.00]\cdot 4$$
$$= 95.84 \text{ megawatts}$$

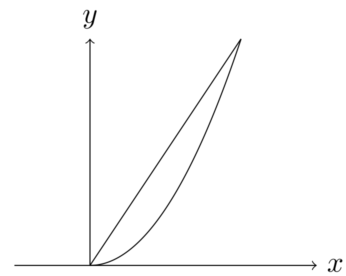
(Make sure your answer includes units.)

3. (a) We first find the points of intersection of these two curves:

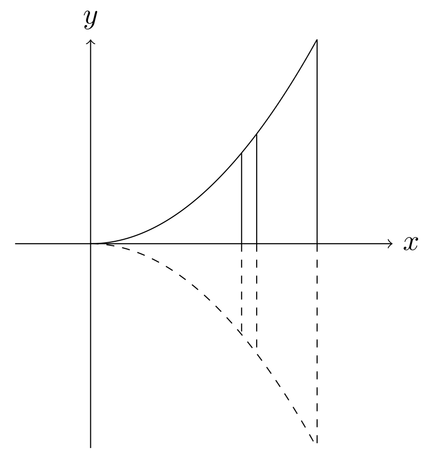
$$6x = 3x^2 \Rightarrow 3x^2 - 6x = 0 \Rightarrow 3x(x-2) = 0 \Rightarrow x = 0 \text{ or } x = 2.$$

A sketch of the two graphs are provided on the right. The area bounded by the curves is thus

$$\int_0^2 (6x - 3x^2)\,dx = 3x^2 - x^3 \Big|_0^2 = 12 - 8 = 4.$$



(b)



We sketch the area getting rotated on the right. Sub-dividing the interval $[0,3]$ into small rectangles with width Δx and the solid is thus divided into discs with volume approximated by $\pi x^2 \Delta x$. Thus, the volume of the solid obtained is

$$\int_0^3 \pi x^2\,dx = \pi \frac{x^3}{3}\Big|_0^3 = 9\pi.$$

(c) The differential equation is separable. Since the initial condition is $y(0) = 1$, we know that $y(t) > 0$ for $t \geq 0$. We rewrite the equation as

$$\frac{1}{y}\frac{dy}{dt} = e^{-t}.$$

Integrating both sides with respect to t gives

$$\int \frac{1}{y}\frac{dy}{dt}\,dt = \int e^{-t}\,dt, \qquad \text{or} \qquad \int \frac{1}{y}\,dy = \int e^{-t}\,dt.$$

Thus,

$$\ln y = -e^{-t} + C.$$

since we know $y > 0$. The initial condition then gives $0 = -1 + C$ so $C = 1$, i.e. $\ln y = 1 - e^{-t}$. Exponentiating both sides gives $y = e^{1-e^{-t}}$.

4. Note that you could cite the p-integral over $[a, \infty)$ and over $[0, a]$ to answer parts (a) - (c) if you so choose but below is a full explanation.

For $a > 0$, consider the improper integral

$$\int_a^\infty \frac{1}{x^p}\,dx = \lim_{R\to\infty}\int_a^R \frac{1}{x^p}\,dx = \lim_{R\to\infty}\begin{cases} \dfrac{1}{-p+1}x^{-p+1}\Big|_a^R & \text{if } p \neq 1 \\ \ln x\Big|_a^R & \text{if } p = 1 \end{cases}$$

$$= \lim_{R\to\infty}\begin{cases} \dfrac{1}{-p+1}\left(R^{-p+1} - a^{-p+1}\right) & \text{if } p \neq 1 \\ \ln R - \ln a & \text{if } p = 1 \end{cases} = \begin{cases} \dfrac{a^{1-p}}{p-1} & \text{if } p > 1 \\ +\infty & \text{if} p \leq 1 \end{cases}.$$

Note that writing $\lim\limits_{x\to a} f(x) = +\infty$ does not mean the limit is infinity because infinity is not a number. All it means is that $f(x)$ grows without bounds.

(a) Using the result above, we conclude that the improper integral does not converge since the value for p in this case is 1.

(b) In this case, we conclude that the improper integral converges (to 1, in fact) since the p-value in this case is 2.

(c) This is the type of improper integral where the integrand is unbounded. Observe that

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+}\frac{-1}{x}\Big|_t^1 = \lim_{t\to 0^+}\left(-1 + \frac{1}{t}\right) = +\infty$$

so the improper integral diverges.

(d) One option is to use the usual definition of improper integral where the limit of integration is unbounded here or use the following argument.

Recall from Calculus I that e^x grows faster than any x^n for any natural number n. Thus, there exists a positive number K such that $e^x \geq x^2 > 0$ for all $x \geq K$. Hence, $e^{-x} = 0 < \dfrac{1}{e^x} \leq \dfrac{1}{x^2}$ for all $x \geq K$. Since the improper integral $\displaystyle\int_K^\infty \frac{1}{x^2}\,dx$ converges, by the comparison test, the improper integral $\displaystyle\int_K^\infty e^{-x}\,dx$ converges. Consequently, the improper integral $\displaystyle\int_1^\infty e^{-x}\,dx$ converges.

5. (a) Let θ be the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$. Then,

$$\cos\theta = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\|\boldsymbol{v}\|\|\boldsymbol{w}\|} = \frac{2}{\sqrt{14}\cdot\sqrt{2}} = \frac{1}{\sqrt{7}}.$$

Thus, $\theta = \cos^{-1}\dfrac{1}{\sqrt{7}} \approx 1.18$ radians.

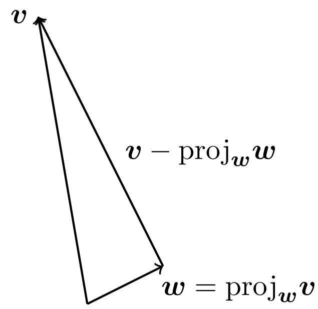
(b) The orthogonal projection of $\boldsymbol{v}$ onto $\boldsymbol{w}$ is

$$\text{proj}_{\boldsymbol{w}}\boldsymbol{v} = \frac{\boldsymbol{v}\cdot\boldsymbol{w}}{\boldsymbol{w}\cdot\boldsymbol{w}}\boldsymbol{w} = \boldsymbol{w} = \langle -1, 0, 1\rangle.$$

The orthogonal decomposition of $\boldsymbol{v}$ is

$$\boldsymbol{v} = \langle -1, 0, 1\rangle + \langle 2, 2, 2\rangle.$$

The figure below gives a sketch of this decomposition:



6. (a) To see if the plane and the line intersect, we determine if there exists a value t such that the equation for $\boldsymbol{r}(t)$ satisfies the plane equation:

$$7 = 3(1 - 2t) - 2 + 2(1 + t) = 3 - 4t \Rightarrow t = -1.$$

Thus, the plane and the line intersect at the point $(3, 2, 0)$.

(b) A normal vector for the plane $x + y = 3$ is $\langle 1, 1, 0\rangle$, and a normal vector to the plane $x + 2y - z = 4$ is $\langle 1, 2, -1\rangle$. Thus, a normal vector for a plane perpendicular to both of the preceding planes is

$$\langle 1, 1, 0\rangle \times \langle 1, 2, -1\rangle = \langle -1, 1, 1\rangle.$$

An equation for such a plane is then $-x + y + z = 0$.

(Note that the constant on the right hand side of the equation for the plane above can be chosen arbitrarily.)

(c) We compute the gradient of the function $f(x, y)$ at P:

$$\nabla f(-1, 2) = \langle 2, 8y\rangle_{(-1,2)} = \langle 2, 16\rangle.$$

Also, $f(-1, 2) = -2 + 16 = 14$. Thus, an equation of the tangent plane to the graph of $f(x, y)$ at P is

$$z - 14 = 2(x + 1) + 16(y - 2),$$

which simplifies to $2x + 16y - z = 16$.

7. Note that $\nabla f(x, y) = \langle 2xe^{-x} - x^2e^{-x}, 2ye^{-x}\rangle = e^{-x}\langle 2x - x^2, 2y\rangle$. Since ∇f is defined everywhere, to find critical points of f, we set

$$\langle 0, 0\rangle = \nabla f(x, y) = e^{-x}\langle 2x - x^2, 2y\rangle$$

and get $x = 0, 2$ and $y = 0$ since $e^{-x} > 0$ for all x.

To determine the nature of these critical points, we need to compute the discriminant $D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - f_{xy}^2(a, b)$ so we first find the second-order partial derivatives:

$$\begin{aligned}
f_{xx}(x, y) &= e^{-x}(2 - 2x) - e^{-x}(2x - x^2) = e^{-x}(2 - 4x + x^2),\\
f_{xy}(x, y) &= 0\\
f_{yy}(x, y) &= 2e^{-x}.
\end{aligned}$$

Then, $D(0, 0) = 4, D(2, 0) = -2e^{-2} \cdot 2e^{-2} = -4e^{-4}$, and $f_{xx}(0, 0) = 2$. Since $D(0, 0) > 0$ and $f_{xx}(0, 0) > 0$, we conclude that $f(0, 0) = 0$ is a local minimum. Since $D(2, 0) < 0$, we conclude that $f(2, 0) = 4e^{-2}$ is a saddle point.

Sample Calculus III Questions

A student need not get 100% on the questions below in order to skip Calculus III (Math 204) and go straight into Linear Algebra (Math 220) or Number Theory (Math 251). An initial score of at least 50% is good enough as long as you feel that a little review would bring your score up to at least 80%. If you feel that you would need more than a little review, then you ought to enroll in Math 204.

1. **Series**

   (a) Which of the following limits equals the sum of the series $\sum_{n=0}^{\infty} a_n$?

   i. $\lim_{n\to\infty} \sum_{n=0}^{\infty} a_n$,

   ii. $\lim_{k\to\infty} \sum_{n=0}^{k} a_n$,

   iii. $\lim_{k\to 0} \sum_{n=0}^{\infty} a_n$,

   iv. $\lim_{n\to\infty} a_n$.

   (b) To what value do the following series converge?

   i. $\sum_{k=2}^{\infty} 5\left(\frac{-2}{3}\right)^k$,

   ii. $\sum_{k=2}^{\infty} \frac{2}{(n-1)(n+1)}$.

   (c) Which of the following series converge?

   i. $\sum_{k=0}^{\infty} 2(3k)$,

   ii. $\sum_{k=0}^{\infty} 2\left(\frac{3}{5}\right)^k$,

   iii. $\sum_{k=1}^{\infty} \frac{(-3)^n}{n!}$,

   iv. $\sum_{k=1}^{\infty} \frac{2}{k^2+1}$.

2. **Taylor Series**

   (a) The Taylor series for $f(x) = e^x$ about $x = 0$ is

   $$1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$$

   i. What is the interval of convergence of the Taylor series for e^x?

   ii. Use the first 4 terms of the Taylor series for e^x to estimate the value of e^2 to three decimal places.

   (b) The Taylor series for $f(x) = \frac{1}{1-x}$ about $x = 0$ is

   $$1 + x + x^2 + x^3 + x^4 + \cdots$$

   with an interval of convergence of $(-1, 1)$.

   i. Find the first 4 non-zero terms of the Taylor series for $f(x) = \frac{1}{1-2x}$.

   ii. What is the interval of convergence for the Taylor series in the problem above?

3. **Multivariable Integration**

   (a) Compute $\displaystyle\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1}\,dx\,dy$.

   (b) Find the length of the path given by $(\sin\theta - \theta\cos\theta, \cos\theta + \theta\sin\theta)$ over the interval $0 \le \theta \le 2$.

   (c) Let $\mathcal{W}$ be the region bounded by $z = 1 - y^2, y = x^2$, and the plane $z = 0$. Calculate the volume of $\mathcal{W}$ as a triple integral in the order $dz\,dy\,dx$.

4. **Change of Variables**

   (a) Let $\mathcal{D}$ be the parallelogram with vertices $(0,0), (3,4), (8,5), (5,1)$. Evaluate $\displaystyle\iint_{\mathcal{D}} xy\,dx\,dy$. Hint: Consider the map $G(u,v) = (5u+3v, u+4v)$.

   (b) Find the volume of the region bounded below by the plane $z = 1$ and above by the sphere $x^2 + y^2 + z^2 = 4$.

5. **Line Integrals**

   (a) Compute $\displaystyle\int_{\mathcal{C}} x^2 z\,ds$, where $\mathcal{C}$ is the path parametrized by $\boldsymbol{r}(t) = \langle e^t, \sqrt{2}t, e^{-t}\rangle$ for $0 \le t \le 1$.

   (b) Evaluate $\displaystyle\int_{\mathcal{C}} z\,dx + x^2\,dy + y\,dz$, where $\mathcal{C}$ is parametrized by $\boldsymbol{r}(t) = \langle \cos t, \tan t, t\rangle$ for $0 \le t \le \dfrac{\pi}{4}$.

6. **Conservative Vector Fields**

   (a) Determine if the field $\boldsymbol{F} = \langle yze^{xy}, xze^{xy}, e^{xy}\rangle$ is conservative. If it is, find a potential function for $\boldsymbol{F}$.

   (b) Evaluate $\displaystyle\int_{\mathcal{C}} yze^{xy}\,dx + (xze^{xy} - z)\,dy + (e^{xy} - y)\,dz$, where $\mathcal{C}$ is the line segment going from $(0,2,0)$ to $(1,0,1)$.

Calculus III Answer Key

Make sure you not only get the correct answers but also provide sufficient justification.

1. (a) By definition, the sum of a series is the limit of the sequence of its partial sum:

$$\sum_{n=0}^{\infty} = \lim_{k\to\infty} S_k,$$

where $S_k = a_0 + a_1 + \cdots + a_k = \sum_{n=0}^{k} a_n$. Thus, the answer is ii.

(b) i. This is a geometric series with $r = -2/3$, which satisfies $|r| < 1$, so we have

$$\sum_{k=2}^{\infty} 5\left(\frac{-2}{3}\right)^k = \frac{\text{first term}}{1-r} = \frac{5\cdot(-2/3)^2}{1-(-2/3)} = \frac{20/9}{5/3} = \frac{4}{3}.$$

ii. Let $a_n = \dfrac{2}{(n-1)(n+1)}$. First, observe that

$$\frac{2}{(n-1)(n+1)} = \frac{n+1-(n-1)}{(n-1)(n+1)} = \frac{n+1}{(n-1)(n+1)} - \frac{n-1}{(n-1)(n+1)} = \frac{1}{n-1} - \frac{1}{n+1}.$$

(You could also use the method of partial fractions to obtain the difference of fractions above.)
We compute some partial sums

$$\begin{aligned}
S_2 &= a_2 = \frac{1}{1} - \frac{1}{3},\\
S_3 &= a_2 + a_3 = \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4},\\
S_4 &= a_2 + a_3 + a_4 = \frac{1}{1} - \frac{1}{3} + \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{5} = 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5},\\
&\cdots\\
S_N &= a_2 + \cdots + a_N = 1 + \frac{1}{2} - \frac{1}{N} - \frac{1}{N+1}.
\end{aligned}$$

Hence,

$$\sum_{k=2}^{\infty} \frac{2}{n^2-1} = \lim_{N\to\infty} S_N = \lim_{N\to\infty} 1 + \frac{1}{2} - \frac{1}{N} - \frac{1}{N+1} = \frac{3}{2}.$$

(c) Which of the following series converge?

i. Let $a_k = 6k$. Since $\lim_{k\to\infty} a_k = \infty$, by the divergence test, $\sum_{k=0}^{\infty} 2(3k)$ diverges.

ii. The series $\sum_{k=0}^{\infty} 2\left(\frac{3}{5}\right)^k$ is a geometric series with $r = 3/5$, which satisfies $|r| < 1$ so the series converges.

iii. Let $a_n = \dfrac{(-3)^n}{n!}$. Note that

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\frac{(-3)^{n+1}}{(n+1)!}}{\frac{(-3)^n}{n!}}\right| = \frac{3}{n+1}$$

so $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 0 < 1$. The series $\sum_{k=1}^{\infty}\frac{(-3)^n}{n!}$ converges absolutely and thus converges.

iv. Observe that $0 < \frac{2}{k^2+1} < \frac{2}{k^2}$ for all $k \geq 1$. The series $\sum_{k=1}^{\infty}\frac{2}{k^2}$, which is a p-series with $p = 2 > 1$, converges. By the direct comparison test, the series $\sum_{k=1}^{\infty}\frac{2}{k^2+1}$ converges.

2. **Taylor Series**

(a) i. Set $a_n = \frac{x^n}{n!}$. Then,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x|}{n+1} = 0$$

so the interval of convergence of the Taylor series for e^x is $(-\infty,\infty)$. That is, the Taylor series converges to e^x for all x.

ii. Plugging in $x = 2$, we get

$$e^2 \approx 1 + 2 + \frac{2^2}{2} + \frac{2^3}{6} \approx 6.333$$

(b) i. Replace x by $2x$ in the Taylor series above to get the first 4 non-zero terms of the Taylor series for $f(x) = \frac{1}{1-2x}$ as

$$1 + 2x + 4x^2 + 8x^3.$$

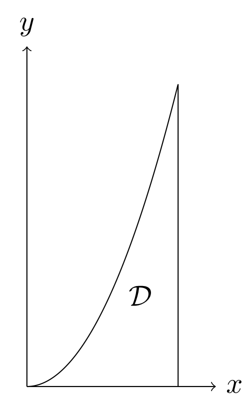
ii. Solving $-1 < 2x < 1$ gives the interval of convergence for the Taylor series for $f(x)$ as $(-0.5, 0.5)$.

3. **Multivariable Integration**

(a)

We sketch $\mathcal{D}$, the region of integration on the right. Then,

$$\begin{aligned}\int_0^4\int_{\sqrt{y}}^2 \sqrt{x^3+1}\,dx\,dy &= \int_0^2\int_0^{x^2}\sqrt{x^3+1}\,dy\,dx\\ &= \int_0^2 y\sqrt{x^3+1}\Big|_{y=0}^{y=x^2}\,dx\\ &= \int_0^2 x^2\sqrt{x^3+1}\,dx\\ &= \frac{1}{3}\cdot\frac{2}{3}(x^3+1)^{3/2}\Big|_0^2 = \frac{52}{9}.\end{aligned}$$



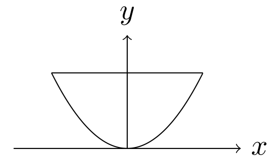
(b) We have $x'(\theta) = \cos\theta - \cos\theta + \theta\sin\theta = \theta\sin\theta$, and $y'(\theta) = -\sin\theta + \sin\theta + \theta\cos\theta = \theta\cos\theta$ so

$$\|\boldsymbol{r}'(t)\| = \sqrt{x'(\theta)^2 + y'(\theta)^2} = \sqrt{\theta^2\sin^2\theta + \theta^2\cos^2\theta} = \sqrt{\theta^2} = |\theta| = \theta$$

since $\theta \in [0,2]$. The length of the path is thus

$$s = \int_0^2 \|\boldsymbol{r}'(t)\|\,dt = \int_0^2 \theta\,d\theta = 2.$$

(c)



The projection of $\mathcal{W}$ onto the xy-plane, $\mathcal{D}$ is given on the right. Notice that $\mathcal{D} = \{(x,y) : -1 \le x \le 1, x^2 \le y \le 1\}$.

Hence, the volume of $\mathcal{W}$ is

$$\begin{aligned} V(\mathcal{W}) &= \int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y^1} 1\, dz\, dy\, dx = \int_{-1}^{1}\int_{x^2}^{1} z\big|_0^{1-y^2}\, dy\, dx \\ &= \int_{-1}^{1}\int_{x^2}^{1} (1-y^2)\, dy\, dx = \int_{-1}^{1} (y - \frac{1}{3}y^3)\bigg|_{x^2}^{1} dx \\ &= \int_{-1}^{1} \left(\frac{2}{3} - x^2 + \frac{1}{3}x^6\right) dx = \frac{2}{3}x - \frac{1}{3}x^3 + \frac{1}{21}x^7\bigg|_{-1}^{1} = \frac{16}{21}. \end{aligned}$$

4. **Change of Variables; Other Coordinate Systems**

(a) The lines that form the boundary of the parallelogram have equations $4x-3y=0, 4x-3y=17, 5y-x=0, 5y-x=17$, i.e.

$$\mathcal{D} = \{(x,y) : 0 \le 4x-3y \le 17, 0 \le 5y-x \le 17\}.$$

Consider the map $G(u,v) = (5u+3v, u+4v)$. Then, $x = 5u+3v$ and $y = u+4v$ so $4x-3y = 17u$ and $5y-x = 17v$. Thus,

$$\mathcal{D}_0 = \{(u,v) : 0 \le u \le 1, 0 \le v \le 1\}.$$

Now,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 5 & 3 \\ 1 & 4 \end{vmatrix} = 17.$$

The,

$$\begin{aligned} \iint_{\mathcal{D}} xy\, dx\, dy &= \iint_{\mathcal{D}_0} x(u,v)y(u,v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv \\ &= 17\int_0^1\int_0^1 (5u+3v)(u+4v)\, du\, dv \\ &= 17\int_0^1\int_0^1 5u^2 + 23uv + 12v^2\, du\, dv \\ &= 17\int_0^1 \left(\frac{5}{3}u^3 + \frac{23}{2}u^2v + 12v^2u\right)\bigg|_{u=0}^{u=1} dv \\ &= 17\int_0^1 \left(\frac{5}{3} + \frac{23}{2}v + 12v^2\right) dv = 17\left(\frac{5}{3}v + \frac{23}{4}v^2 + 4v^3\right)\bigg|_0^1 = \frac{2329}{12}. \end{aligned}$$

(b) We use spherical coordinates for this computation. The equation of the plane $z=1$ in polar coordinates is $\rho\cos\phi = 1$, and the equation for the sphere is $\rho = 2$. Where the plane and the sphere intersect, we have $x^2+y^2=3$ and $z=1$ so $\phi_0=\tan^{-1}\sqrt{3}=\pi/3$. The volume of the region is

$$\begin{aligned}
\iiint 1\,dV &= \int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\
&= 2\pi\int_0^{\pi/3}\frac{1}{3}\rho^3\sin\phi\Big|_{\rho=\sec\phi}^{\rho=2}\,d\phi = \frac{2\pi}{3}\int_0^{\pi/3}\sin\phi(8-\sec^3\phi)\,d\phi \\
&= \frac{2\pi}{3}\int_0^{\pi/3}\left(8\sin\phi-\tan\phi\sec^2\phi\right)\,d\phi \\
&= \frac{2\pi}{3}\left(-8\cos\phi-\frac{1}{2}\tan^2\phi\right)\Big|_0^{\pi/3} = \frac{5\pi}{3}.
\end{aligned}$$

5. **Line Integrals**

   (a) We first compute

   $$\|\boldsymbol{r}'(t)\| = \|\langle e^t,\sqrt{2},-e^{-t}\rangle\| = \sqrt{e^{2t}+2+e^{-2t}} = \sqrt{(e^t+e^{-t})^2} = |e^t+e^{-t}| = e^t+e^{-t}$$

   since $e^t+e^{-t}>0$ for all t. Then,

   $$\int_{\mathcal{C}} x^2z\,ds = \int_0^1 (e^t)^2(e^{-t})(e^t+e^{-t})\,dt = \int_0^1 (e^{2t}+1)\,dt = \frac{1}{2}e^{2t}+t\Big|_0^1 = \frac{1}{2}(e^2+1).$$

   (b) Since $\boldsymbol{r}'(t)=\langle -\sin t,\sec^2 t,1\rangle$, we have

   $$\begin{aligned}
   \int_{\mathcal{C}} z\,dx+x^2\,dy+y\,dz &= \int_0^{\pi/4}\langle t,\cos^2 t,\tan t\rangle\cdot\langle -\sin t,\sec^2 t,1\rangle\,dt \\
   &= \int_0^{\pi/4} -t\sin t+1+\tan t\,dt \\
   &= t\cos t-\sin t+t+\ln|\sec t|\Big|_0^{\pi/4} = \frac{\pi}{4\sqrt{2}}-\frac{1}{\sqrt{2}}+\frac{\pi}{4}+\ln\sqrt{2}.
   \end{aligned}$$

6. **Conservative Vector Fields**

   (a) Let $f(x,y,z)=ze^{xy}$. Then, $\nabla f=\langle yze^{xy},xze^{xy},e^{xy}\rangle=\boldsymbol{F}$ so $\boldsymbol{F}$ is conservative and f, as defined, is a potential function for $\boldsymbol{F}$.

   (b) Let $\boldsymbol{F}$ be the field as in part (a), and $\boldsymbol{G}(x,y,z)=\langle 0,-z,-y\rangle$. Then,

   $$\int_{\mathcal{C}} yze^{xy}\,dx+(xze^{xy}-z)\,dy+(e^{xy}-y)\,dz = \int_{\mathcal{C}}\boldsymbol{F}\cdot d\boldsymbol{r}+\int_{\mathcal{C}}\boldsymbol{G}\cdot d\boldsymbol{r}.$$

   Since $\boldsymbol{F}$ is conservative, for f defined in part (a), by the fundamental theorem for conservative vector fields, we have

   $$\int_{\mathcal{C}}\boldsymbol{F}\cdot d\boldsymbol{r} = f(1,0,1)-f(0,2,0)=1.$$

   Parametrize the line segment by $\boldsymbol{r}(t)=\langle 0,2,0\rangle+t\langle 1,-2,1\rangle$ for $0\le t\le 1$ so that $\boldsymbol{r}'(t)=\langle 1,-2,1\rangle$. Then,

   $$\int_{\mathcal{C}}\boldsymbol{G}\cdot d\boldsymbol{r} = \int_0^1\langle 0,-t,2t-2\rangle\cdot\langle 1,-2,1\rangle\,dt = \int_0^1(4t-2)\,dt = 0.$$

   Overall, we get $\displaystyle\int_{\mathcal{C}} yze^{xy}\,dx+(xze^{xy}-z)\,dy+(e^{xy}-y)\,dz=1.$